Continuity

CONTINUITY AT A POINT *A real function $f(x)$ is said to be continuous at a point a of its domain if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.*

Thus, $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

If $f(x)$ is not continuous at a point, it is said to be *discontinuous* at that point.

REMARK $f(x)$ is discontinuous at $x = a$ in each of the following cases:

(i) $f(a)$ is not defined

(ii) $\lim_{x \to a} f(x)$ does not exist

(iii) $\lim_{x \to a} f(x) \neq f(a)$ {*removable discontinuity*}

SOLVED EXAMPLES

EXAMPLE 1 *Show that $f(x) = x^3$ is continuous at $x = 2$.*

SOLUTION We have $f(2) = 2^3 = 8$;

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} (2+h)^3 = \lim_{h \to 0} (8 + h^3 + 12h + 6h^2) = 8;$$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} (2-h)^3 = \lim_{h \to 0} (8 - h^3 - 12h + 6h^2) = 8.$$

$$\therefore \quad \lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = f(2).$$

Hence, $f(x)$ is continuous at $x = 2$.

EXAMPLE 2 *Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.*

SOLUTION We have $f(n) = [n] = n$;

$$\lim_{x \to n+} f(x) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} [n+h] = n \quad \{\because \ [n+h] = n\};$$

$$\lim_{x \to n-} f(x) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} [n-h] = (n-1) \{\because \ [n-h] = (n-1)\}.$$

Thus, $\lim_{x \to n+} f(x) \neq \lim_{x \to n-} f(x)$ and therefore, $\lim_{x \to n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

EXAMPLE 4 *Discuss the continuity of the function $f(x)$ at $x = 0$, if*

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases}$$

[CBSE 2002]

SOLUTION Clearly, $f(0) = (2 \times 0 + 1) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} [2(0 + h) + 1] = \lim_{h \to 0} (2h + 1) = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} [2(0 - h) - 1)] = \lim_{h \to 0} (-2h - 1) = -1.$$

Thus, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 5 *Show that the function* $f(x) = \begin{cases} 3x - 2, & \text{when } x \leq 0 \\ x + 1, & \text{when } x > 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION We have, $f(0) = (3 \times 0 - 2) = -2$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} (h + 1) = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} [3(-h) - 2] = \lim_{h \to 0} (-3h - 2) = -2.$$

$\therefore$ $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 6 *Show that the function* $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

SOLUTION It is being given that $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{-h}{|h|} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

$$\therefore \quad \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x).$$

So, $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 7 *Examine the continuity of the function*

$$f(x) = \begin{cases} \dfrac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \text{ at } x = 0. \end{cases}$$

[CBSE 2004]

SOLUTION We have $f(0) = 1$.

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{|\sin(0+h)|}{(0+h)} = \lim_{h \to 0} \frac{|\sin h|}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{|\sin(-h)|}{-h} = \lim_{h \to 0} \frac{|-\sin h|}{-h} = \lim_{h \to 0} \frac{\sin h}{-h} = -1.$$

$\lim\limits_{x \to 0+} f(x) \neq \lim\limits_{x \to 0} f(x)$. So, $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

EXAMPLE 9 *Let* $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3. \end{cases}$

Show that $f(x)$ is continuous at $x = -3$ but discontinuous at $x = 3$.

SOLUTION We have $f(-3) = |-3| + 3 = (3 + 3) = 6.$

$$\lim_{x \to (-3)+} f(x) = \lim_{h \to 0} f(-3 + h)$$

$$= \lim_{h \to 0} \{-2(-3 + h)\} = \lim_{h \to 0} (6 - 2h) = 6.$$

$$\lim_{x \to (-3)-} f(x) = \lim_{h \to 0} f(-3 - h)$$

$$= \lim_{h \to 0} (|-3 - h| + 3) = \lim_{h \to 0} \{|-(3 + h)| + 3\}$$

$$= \lim_{h \to 0} \{(3 + h) + 3\} = 6.$$

$\therefore \quad \lim_{x \to (-3)+} f(x) = \lim_{x \to (-3)-} f(x) = 6.$ So, $\lim_{x \to (-3)} f(x) = 6.$

Thus, $\lim_{x \to (-3)} f(x) = f(-3) = 6.$

Hence, $f(x)$ is continuous at $x = -3$.

Now, $f(3) = (6 \times 3 + 2) = 20.$

$$\lim_{x \to 3-} f(x) = \lim_{h \to 0} f(3 - h)$$

$$= \lim_{h \to 0} \{-2(3 - h)\} = \lim_{h \to 0} (-6 + 2h) = -6.$$

Thus, $f(3) \ne \lim_{x \to 3-} f(x).$

Hence, $f(x)$ is discontinuous at $x = 3$.